A toolbox for fitting non-separable space-time log-Gaussian Cox models using R-INLA

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Sumário

Burkitt data, splancs (Rowlingson and Diggle 1993)

- Observed time lymphoma cases from year 1960 to 1975 (16 years)

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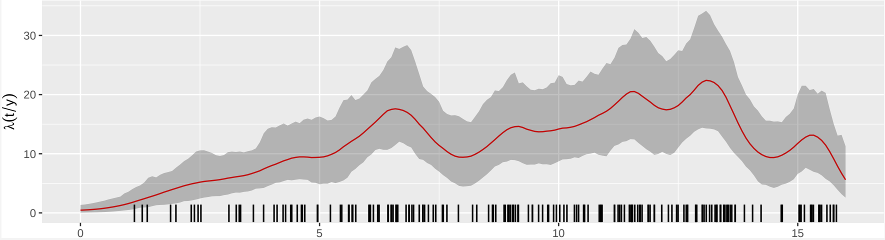
Fitted intensity function and 95% credibility interval



Observed vs expected number of cases

```
sum(dat1$year<5) # Number of observed cases (year < 5)
## [1] 27
head(pred.res, 5)
##    year mean   sd q0.025 median q0.975  smin smax   cv  var
## 1 0.000 0.45 0.39  0.035   0.37    1.3 0.018  1.8 0.86 0.15
## 2 0.083 0.49 0.40  0.043   0.40    1.4 0.021  1.8 0.83 0.16
## 3 0.167 0.53 0.42  0.059   0.45    1.5 0.025  1.8 0.80 0.18
## 4 0.250 0.57 0.44  0.078   0.49    1.6 0.031  1.9 0.76 0.19
## 5 0.333 0.63 0.46  0.092   0.52    1.8 0.043  2.0 0.73 0.21
sum(pred.res$mean[pred.res$year<5]/12) # Expected (year < 5)
## [1] 24
c(nrow(dat1), sum(pred.res$mean)/12) # Total Obs. and Exp.
## [1] 188 183
```

The data and modeling approaches

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Common approach

- Aggregate the data into discrete grid cells
- Model the number of cases in each grid cell t as Poisson(λ_t)
- Model the log(λ_t)
  - pick the **model** framework you like the most

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Point process modeling approach

- Treat the data as it is, no aggregation

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The log-Likelihood for a set of points

- Given the domain $[0, L]$ and $\mathbf{y}$

$$l(\lambda(.)|\mathbf{y}) = \boxed{L} - \log\left(\int_0^L \boxed{\lambda(t)} \partial t\right) + \sum_{i=1}^{n} \log(\lambda(\boxed{t_i}))$$

  - $\boxed{L}$ is the size of the time domain
  - $\boxed{\lambda(t)}$ is the intensity at time t
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Direct likelihood approximation, Simpson et al. (2016)

$$l(\lambda(.)|\mathbf{y}) \approx L - \sum_{j=1}^{m} w_j \log\lambda(t_j) + \sum_{i=1}^{n}\log(\lambda(t_i))$$

- m integration points $t_1, ..., t_m$, weights $w_1, ..., w_m$
  - if the grid is equaly spaced, $w_j = w_0$

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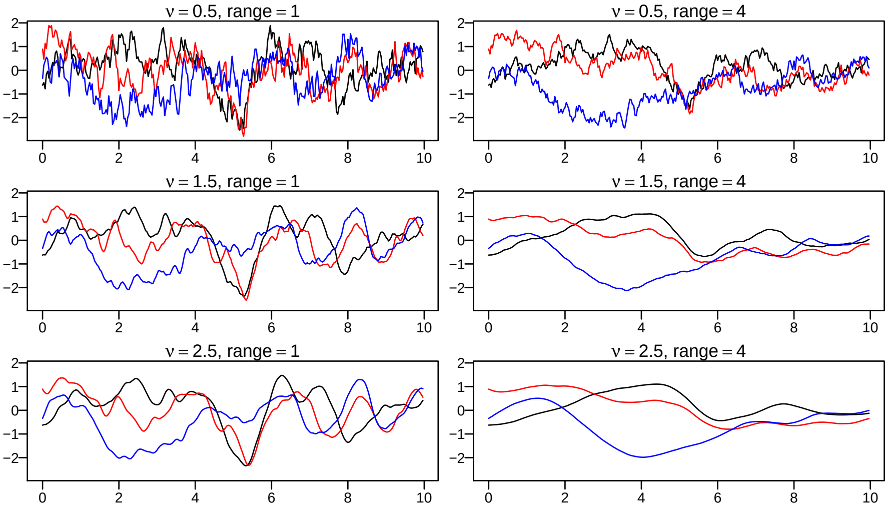
Figure 1: A latent Gaussian model !!!

Matern 1D field samples (smoothness ν and range)



Our model framework: Latent Gaussian models (LGM)

- Bayesian hierarchical model

$\boldsymbol{y} \mid \boldsymbol{x}, \boldsymbol{\theta}$	$\prod_i \pi(y_i \mid \eta_i, \boldsymbol{\theta})$	Several distributions!
$\boldsymbol{x} \mid \boldsymbol{\theta}$	$\pi(\boldsymbol{x} \mid \boldsymbol{\theta}) : \mathcal{N}(\mathbf{0}, \mathbf{Q}(\boldsymbol{\theta})^{-1})$	Gaussian (GMRF)!
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- **INLA**: Integrated Nested Laplace Approximations, Rue, Martino, and Chopin (2009) and Rue et al. (2017)

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The idea into practice, using inlabru

```
# data
dat1 <- data.frame(year = burkitt$t/365.25)
# domain (0, 16) and integration points
mesh.t <- inla.mesh.1d(seq(-3, 20, 1/12))
# assumed GMRF model (Matern) definition
model <- year ~ 0 + Intercept +
  spde1D(map = year,
         model = inla.spde2.pcmatern(
           mesh = mesh.t,
           prior.sigma = c(1, 0.5), # P(sigma>1) = 0.5
           prior.range = c(0.1, 0.01)))# P(range<0.1) = 0.01
# fit the model
fit.model1 <- lgcp(model, dat1)
```

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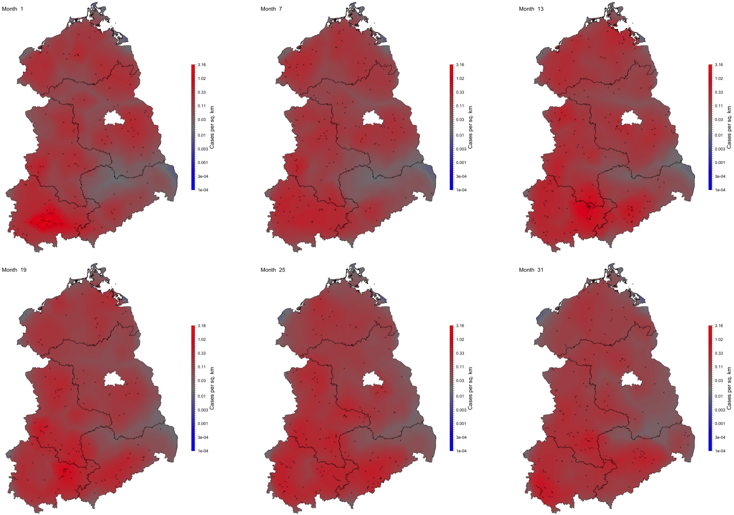
Space-time example: fox rabie cases



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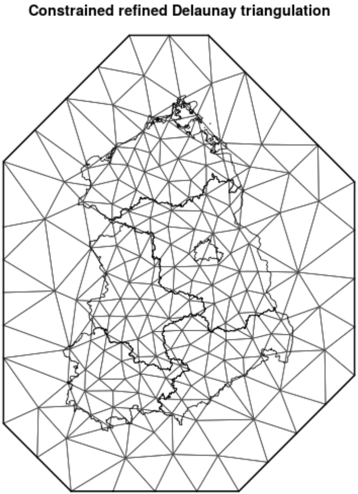
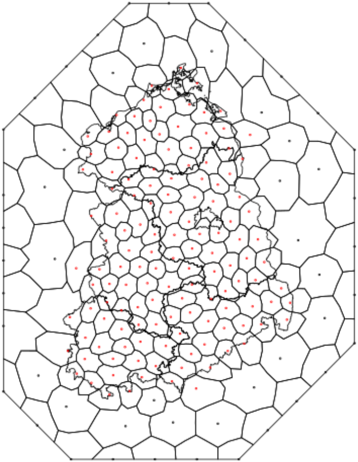
  - separable: time Kronecker space
  - non-separable: **spacetime** driven process
- code!

Adding spatial integration points





Space-time non-separable process

- Consider the stochastic iterated heat equation, Krainski (2018)

$$\left(\gamma_t \frac{d}{dt} + L^{\alpha_s/2}\right)^{\alpha_t} u(t) = \dot{W}_{\gamma_e^2 L^{\alpha_e}}(t).$$

- differential (system of) equations with respect to time
- L: Spatial Matern
- smoothness parameters $(\alpha_t, \alpha_s, \alpha_e)$
  - flexible model, consider some cases
- scale parameters $(\gamma_t, \gamma_s, \gamma_e) > 0$

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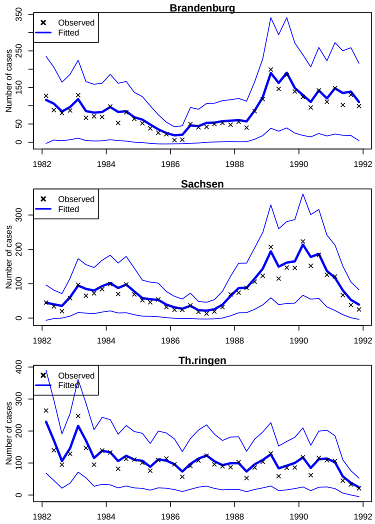
Advantage of using SPDEs

- sparse precision matrix representation!

Code

- consider the INLA package, Rue et al. (2017)
  - define the precision matrix using the `rgeneric` model
- consider the efficient PARDISO library, Kourounis, Fuchs, and Schenk (2018)

Preliminar result: Observed and expected cases, big areas per time



Work in progress

- wrap it into a function for the user
  - dozens lines of code write now
- parameter interpretation $\rightarrow$ define new ones such as
  - spatial range, temporal range, marginal variance

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